Theory of Convective Heat and Mass Transfer to Spherical-Cap Bubbles

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The forced convective heat and mass transfer to spherical-cap bubbles were investigated analytically. Potential flow was assumed outside the bubble except at the bubble surface where the boundary layer approximation was adopted. The present analytical study produced the following exact solution:

$$Nu = 2.113\ (Pe)^{1/2}$$

for the average Nusselt number where Pe is the Peclet number. The derived equation was compared with the available experimental mass-transfer results.

Introduction

The motion and effects of bubbles are of great importance in many modern industrial applications where heat or mass transfer are involved. Examples of these applications are: flow of immiscible fluids in vertical passages or tubes, fluidized beds, absorption of gases in liquid columns, fermentation, agitation, stirring, sewage purification processes, direct contact heat exchangers, and so on.

Bubbles assume various shapes during their motion; they nucleate into spherical form, then into ellipsoidal shapes and finally reach a fairly stable shape of a spherical cap when certain conditions (Table 1) are satisfied. These changes of shape can be seen by the naked eye when bubbles are allowed to ascend in a transparent liquid column.

M in Table 1 represents Morton number ($M = g\mu^4/\rho\sigma^3$). Liquids like water, methyl alcohol, turpentine, and so on are considered as low M fluids (Moore, 1959).

There seems to be disagreement among authors (Table 2) about the size of spherical-cap bubbles. The concept of an equivalent diameter (d) is used widely for sizing the spherical-cap bubble, and it is the diameter of a sphere having the same volume as the bubble.

An interesting feature of the flow dynamics of the spherical-cap bubble is that its rise in liquids occurs independently of the properties of the liquid. Therefore, the velocities of spherical-cap bubbles were found by Davies and Taylor (1950) to be related to their apparent radii of curvature (a) through a relation of the form:

$$u = B(ga)^{1/2} \tag{1}$$

where g is the gravitational acceleration and B is a constant equal to 2/3. Collins (1966) has modified this constant to become equal to 0.652 which is in slightly better accord with experimental values. Haberman and Morton (1956) obtained an almost identical result but expressed the velocity in terms of the equivalent radius (r_e) as:

Table 1 Conditions of the Spherical-Cap Bubble

Author	Conditions	
Ryskin and Leal (1984)	$We \geq 30$	
Bhaga and Weber (1981)	$Re \geq 45$	
Grace (1970)	$Eo \geq 40$	
Wegener and Parlange (1973)	$C_D = 2.7$	Low M fluids
Harper (1972)	$C_D = 1.95$	($M < 10^{-8}$)

Table 2 Size of Spherical-Cap Bubble

Author	Equiv. Dia. d (cm)
Saffman (1956)	2.0
Uno and Kintter (1956)	1.8
Davies and Taylor (1950)	1.8
Haberman and Morton (1956)	1.7
Baker and Chao (1965)	1.0

$$u = 1.02(gr_e)^{1/2} \quad (2)$$

The observations of various investigators revealed that the angle θ_m specifying the edge of the spherical-cap bubble (Figure 1) lies around 50° without any evident variation in the volume of the bubble (Clift et al., 1978). This makes $a = 2.4r_e$ because the volume (H) of the spherical-cap bubble is:

$$H = \pi a^3 \left(\frac{2}{3} - \cos\theta_m + \frac{1}{3}\cos^3\theta_m \right) \quad (3)$$

It was found experimentally (Hnat and Buckmaster, 1976) that the spherical-cap bubble develops, under certain physical conditions, a thin sheet of air called a skirt, trailing from the bubble corners and partially enclosing the wake.

The following review is concerned with the rates of mass transfer to spherical-cap bubbles, as literature on heat transfer to the cap is virtually nonexistent. Baird and Davidson (1962) obtained the following equation which predicts the rates of mass transfer from the frontal spherical portion of the cap as:

$$K_L = 0.975 d^{-1/4} D^{1/2} g^{1/4} \text{ (cm/s)} \quad (4)$$

Lochiel and Calderbank (1964) used the potential flow around the frontal spherical portion of the cap and assumed a thin concentration boundary layer over the surface to get:

$$K_L = 1.03 d^{-1/4} D^{1/2} g^{1/4} \text{ (cm/s)} \quad (5)$$

Lochiel and Calderbank made the above equation dimensionless and compared it with Boussinesq's and Higbie's (1935) mass-transfer equation for single spherical bubble (that is, $Sh_s = 1.13(ScRe)^{1/2}$) to get:

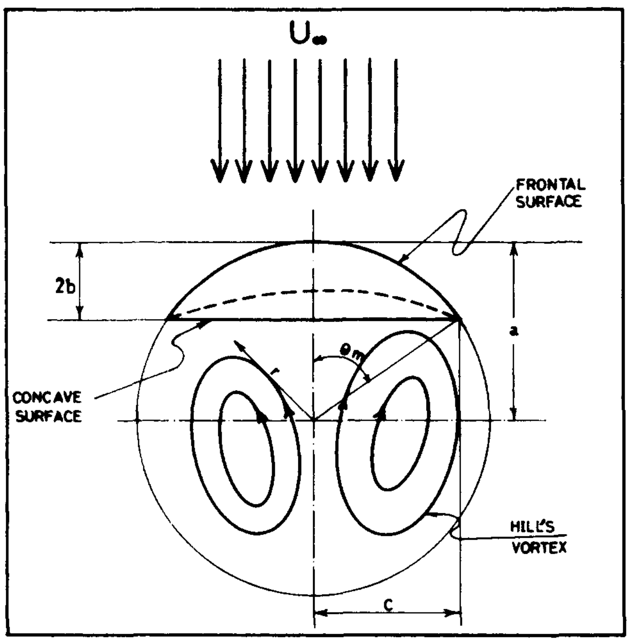

Figure 1. Geometry of spherical-cap bubble.

$$\frac{Sh}{Sh_s} = 1.58 \frac{(3E^2+4)^{2/3}}{E^2+4} \quad (6)$$

where $E = 2C/2b$ is the eccentricity.

Kendoush (1976) adopted the spherical stream function for the frontal part of the cap with the assumption of a thin concentration boundary layer over the frontal spherical surface to obtain:

$$K_L = 1.01 d^{-1/4} D^{1/2} g^{1/4} \text{ (cm/s)} \quad (7)$$

It should be noted that Eq. 2 was used in Eq. 7.

Coppus and Rietema (1981) tried experimentally to isolate the effect of mass transfer from the rear surface of the cap by screening it with aluminum powder. They performed a parametric study based on the surface renewal concept which described well the screening effects.

Brignell (1974) calculated the contact time for the liquid element to traverse a certain distance within the wake of the cap by assuming that the velocity field in the wake can be described by Hill's vortex. The theory predicts a mass-transfer coefficient at the rear of the cap with same order of magnitude as for the front of the cap which is in variance with the present theory. The mass transfer to the skirted spherical-cap bubble was derived by Davenport et al. (1967).

Calderbank et al. (1970) proposed a solution for the mass transfer from the rear of the cap. They adopted cylindrical coordinates for a physical system (that is, the bubble) which is spherical. Naturally, they obtained a Bessel function solution for the equations of motion instead of the usual Legendre's polynomial or the Gaussian error function. They did not use the more appropriate spherical coordinates for the equations of motion (see Carslaw and Jaeger, 1959). However, they derived a formula for the contact time, but Jean and Fan (1990) considered the contact time as an imperfect model on the basis that the presence of a rear stagnation point makes the contact time unrealistically infinite.

Jean and Fan (1990) derived a formula for the mass transfer from the frontal spherical surface of the cap which is analogous to that of Lochiel and Calderbank (1964), and they assumed a potential flow in the wake of the cap to get the rear mass-transfer coefficient. The model of Jean and Fan (1990) compares well with the experimental results of Calderbank et al. (1970).

The main objective of the present work is to provide a complete closed form analytical solution for the heat and mass transfer from both the frontal and the rear surfaces of the spherical-cap bubble.

New Theoretical Analysis

The origin of the spherical system of coordinates is taken at the center of a fixed spherical envelope, the upper part of it is occupied by the spherical-cap bubble, as shown in Figure 1. The flow field of the continuous phase is represented by (U_∞), which is the velocity of the undisturbed continuous medium at infinity. The motion of the continuous phase is symmetrical about the centerline of the spherical cap which is parallel to the flow.

The stream function for the irrotational and inviscid flow field around the frontal spherical part of the cap is assumed to be the same as that over the forward part of a sphere of the same radius (a) (Milne-Thomson, 1972) as:

$$\psi=\frac{1}{2}U_\infty\left(1-\frac{a^3}{r^3}\right)r^2\sin^2\theta \tag{8}$$

with $r\geq a$.

Parlange (1969) developed a theory for the laminar closed wake region of the spherical-cap bubble where the flow was assumed inviscid, vortical, and confined to the spherical Hill's vortex as:

$$\psi=-\frac{3}{4}U_\infty\left(1-\frac{r^2}{a^2}\right)r^2\sin^2\theta \tag{9}$$

with $r\leq a$.

Now it is possible to calculate the velocity components U and V of the flow around the cap from:

$$(\vec{U},\ \vec{V})=-\nabla\psi \tag{10}$$

which gives

$$V=-U_\infty\cos\theta\left(1-\frac{a^3}{r^3}\right) \tag{11}$$

$$U=U_\infty\sin\theta\left(1+\frac{1}{2}\frac{a^3}{r^3}\right) \tag{12}$$

for the frontal spherical surface of the cap and

$$U_r=-\frac{3}{2}U_\infty\sin\theta\left(1-2\frac{r^2}{a^2}\right) \tag{13}$$

$$V_r=\frac{3}{2}U_\infty\cos\theta\left(1-\frac{r^2}{a^2}\right) \tag{14}$$

for the wake region of the cap. It is assumed a thin boundary layer of thickness (δ) is formed around the frontal spherical surface of the cap as shown in Figure 2. The variable y is the radial distance from the cap surface such that $y=0$ at $r=a$ for the frontal spherical surface and $y=0$ at $r=a-2b$ for the rear concave surface of the cap. The variable y extends through the boundary layer such that:

$$\frac{r-a}{a}=\frac{y}{a}\ll 1 \tag{15}$$

at the frontal spherical surface and:

$$\frac{r-(a-2b)}{a-2b}=\frac{y}{a-2b}\ll 1 \tag{16}$$

at the rear concave surface with $2b\ll a$.

Boundary layer separation from the rim of the spherical cap was predicted by Dorrepaal et al. (1976) and confirmed experimentally by Collins (1979). The present writer believes that the separation region at the rim provides a possible explanation

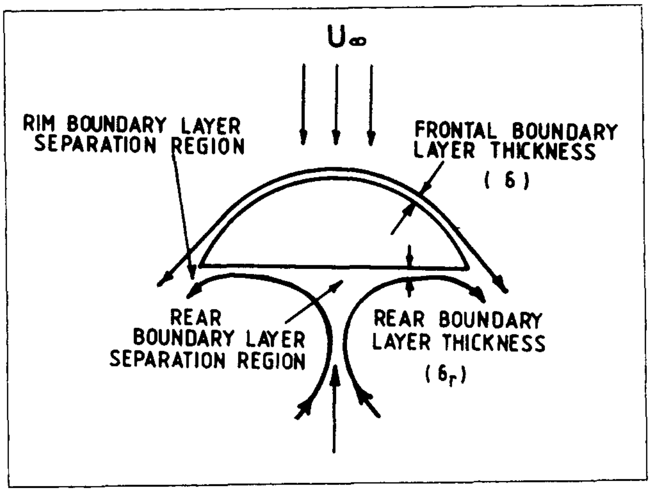

Figure 2. Laminar boundary layer development over frontal and rear surfaces of spherical-cap bubble.

that the skirt formation in certain flow conditions around the spherical-cap bubble originates there. The shear stress within both the frontal and the rear boundary layers tries to entrain a quantity of the gas from the body of the cap to the downstream direction sufficiently enough for the formation of the skirt. However, the appearance of the skirt does not affect the wake structure of the cap.

Utilizing the binomial theorem and retaining the first two terms of:

$$\left(\frac{a}{r}\right)^3=\left(1-\frac{y}{r}\right)^3\cong 1-3\frac{y}{r}-\ \ldots. \tag{17}$$

and

$$\left(\frac{r}{a}\right)^2=\left(1+\frac{y}{a}\right)^2\cong 1+2\frac{y}{a}+\ldots. \tag{18}$$

with the above approximation in Eqs. 11, 12, 13, and 14 we get for the frontal spherical surface of the cap:

$$U\cong\frac{3}{2}U_\infty\sin\theta \tag{19}$$

$$V\cong -3U_\infty\frac{y}{a}\cos\theta \tag{20}$$

and $U_r\cong U$, $V_r\cong V$ for the wake region.

The thermal exchange between the continuous fluid and the concave side of the bubble takes place within the laminar boundary layer that covers the concave surface. If the heat conduction in the angular direction

$$\left(\text{that is, } \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)\right)$$

is small in comparison with the heat conduction in the radial direction

$$\left(\text{that is, } \left(\frac{1}{r}\right)^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right);$$

then the steady-state energy conservation equation in spherical coordinates and with negligible viscous dissipation becomes:

$$V\frac{\partial T}{\partial r}+\frac{U}{r}\frac{\partial T}{\partial \theta}=\alpha\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)\right) \tag{21}$$

The righthand side of the above equation is further approximated on the order of magnitude basis of the boundary layer theory as follows:

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}\cong\frac{\partial^2 T}{\partial r^2} \tag{22}$$

$$\frac{1}{\delta^2}+\frac{2}{a\delta}\cong\frac{1}{\delta^2} \tag{23}$$

The thin boundary layer assumption of Eqs. 17 and 18 allows us to say:

$$\frac{U}{r}\frac{\partial T}{\partial \theta}\cong\frac{U}{a}\frac{\partial T}{\partial \theta} \tag{24}$$

Substituting Eqs. 19, 20, 22, and 24 into Eq. 21 after it gets modified by Eqs. 22 and 24 yields:

$$-3U_\infty\frac{y}{a}\cos\theta\frac{\partial T}{\partial r}+\frac{3}{2}\frac{U_\infty}{a}\sin\theta\frac{\partial T}{\partial \theta}=\alpha\frac{\partial^2 T}{\partial r^2} \tag{25}$$

or

$$\sin\theta\frac{\partial T}{\partial \theta}-2y\cos\theta\frac{\partial T}{\partial y}=\lambda\frac{\partial^2 T}{\partial y^2} \tag{26}$$

where

$$\lambda=\frac{2a\alpha}{3U_\infty} \tag{27}$$

The boundary conditions are (noting that $\theta_m=50°$ or 0.873 rad):

$$T=0 \quad \text{at} \quad y=\infty \quad \text{and} \quad 0.873\geq\theta\geq 0 \tag{28}$$

$$T=T_a \quad \text{at} \quad y=0 \quad \text{and} \quad 0.873\geq\theta\geq 0 \tag{29}$$

The independent variables y and θ are transformed into the new independent variables respectively:

$$\omega=y\sin^2\theta \tag{30}$$

$$\phi=\frac{1}{3}\cos^3\theta-\cos\theta+\frac{2}{3} \tag{31}$$

The above independent variables change Eq. 26 into:

$$\frac{\partial T}{\partial \phi}=\lambda\frac{\partial^2 T}{\partial \omega^2} \tag{32}$$

The new boundary conditions are:

$$T=0 \quad \text{at} \quad \omega=\infty \quad \text{and} \quad 0.1124\geq\phi\geq 0 \tag{33}$$

$$T=T_a \quad \text{at} \quad \omega=0 \quad \text{and} \quad 0.1124\geq\phi\geq 0 \tag{34}$$

The solution of Eq. 32 subject to the above boundary conditions is (Carslaw and Jaeger, 1959):

$$T=T_a\,\mathrm{erfc}\left(\frac{\omega}{2(\lambda\phi)^{0.5}}\right) \tag{35}$$

where

$$\mathrm{erfc}(x)=1-\mathrm{erf}(x)=1-\frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt \tag{36}$$

Substituting the values of ω and ϕ into Eq. 35 gives the complete angular and radial distributions of fluid temperature over the surfaces of the spherical-cap bubble as:

$$T=T_a\mathrm{erfc}\left[\frac{y\sin^2\theta}{2\left(\lambda\left[\frac{1}{3}\cos^3\theta-\cos\theta+\frac{2}{3}\right]\right)^{0.5}}\right] \tag{37}$$

The local convective rate of heat transfer per unit area of the surface of the cap is obtained as follows:

$$\dot{q}_\theta=-K\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{38}$$

$$\dot{q}_\theta=-K\left[\left(\frac{\partial T}{\partial \omega}\right)\left(\frac{\partial \omega}{\partial y}\right)\right]_{y=0} \tag{39}$$

$$\dot{q}_\theta=\frac{KT_a\sin^2\theta}{(\pi\lambda\phi)^{0.5}} \tag{40}$$

however, $\dot{q}_\theta=h_\theta T_a$, therefore, the local Nusselt number is given by:

$$Nu(\theta)=1.693\frac{\sin^2\theta}{(\cos^3\theta-3\cos\theta+2)^{0.5}}(\mathrm{Pe})^{0.5} \tag{41}$$

The above equation is illustrated graphically in Figure 3. It should be noted that the above equation applies for both the convex and the concave surfaces of the cap as shown (not to scale) in Figure 4.

The average convective heat transfer per unit area of the frontal spherical surface of the cap is obtained on the basis of averaging the variation of the local heat-transfer rate within the boundary layer, namely:

$$\dot{q}=\frac{1}{A}\int_0^{\theta_m}\dot{q}_\theta dA \tag{42}$$

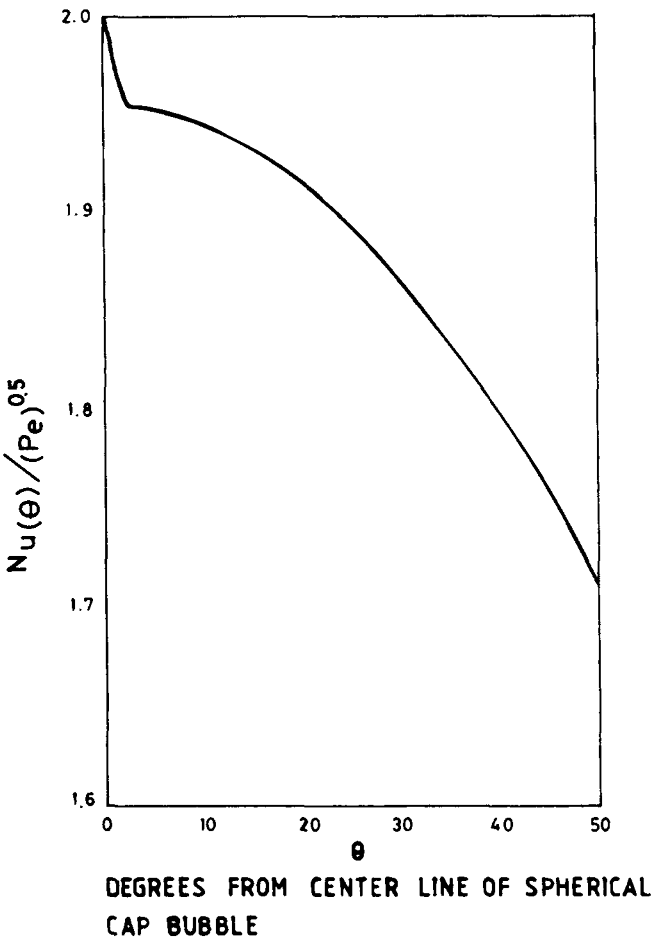

Figure 3. Theoretical prediction of local rate of heat transfer.

where $A = 4\pi ab$ and since $2b = a(1 - \cos\theta_m)$ hence $dA = 2\pi a^2 \sin\theta d\theta$ and therefore:

$$\dot{q} = \frac{aKT_a}{b\sqrt{\pi\lambda}} f(\theta_m) \tag{43}$$

where

$$f(\theta_m) = [\cos^3\theta_m - 3\cos\theta_m + 2]^{0.5} \tag{44}$$

and consequently, the average Nusselt number for the frontal spherical surface of the cap becomes:

$$Nu = 0.564\,\frac{a}{b}\,(Pe)^{0.5} f(\theta_m) \tag{45}$$

when $a = b$ and $\theta_m = \pi$, the spherical-cap approaches the shape of the sphere and the above equation reduces to the single bubble equation of Boussinesq and Higbie (1935) [that is, $(Nu)_s = 1.13\,(Pe)^{0.5}$], and in terms of eccentricity, the above equation becomes after some manipulation

$$Nu = 1.13(Pe)^{0.5}\left(\frac{3E^2 + 4}{E^2 + 4}\right)^{0.5} \tag{46}$$

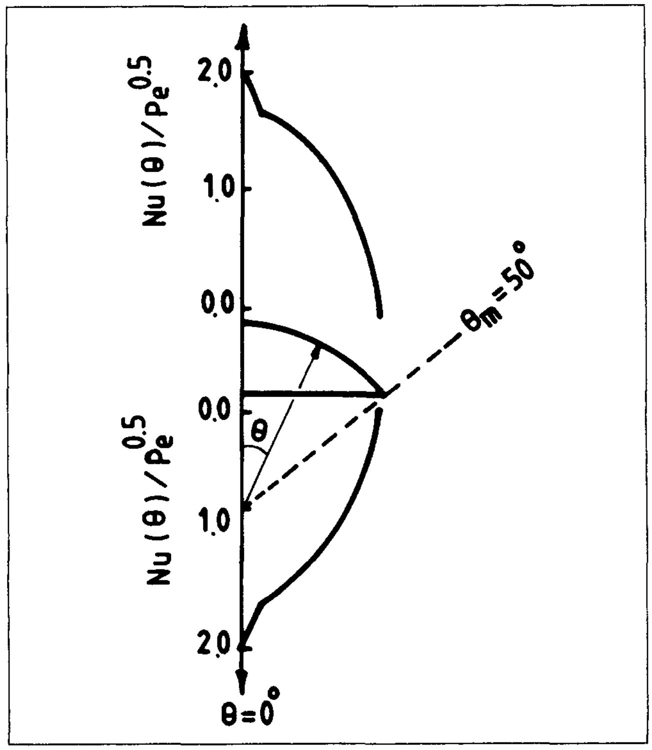

Figure 4. Heat-transfer distribution on all surfaces of cap.

The above equation was derived by Kendoush (1976). For $\theta_m = 50°$, Eq. 45 becomes:

$$Nu = 1.8424\,(Pe)^{0.5} \tag{47}$$

The average convective rate of heat transfer per unit area of the rear surface of the cap is obtained as:

$$\dot{q}_r = \frac{1}{\pi c^2}\int_0^{\theta_m} \dot{q}_\theta \pi a^2 (\cos\theta_m)^2 \sin\theta d\theta \tag{48}$$

with $\theta_m = 50°$, the above equation becomes:

$$\dot{q}_r = 0.156KT_a/\sqrt{\lambda} \tag{49}$$

and the average Nusselt number for the rear surface of the cap is:

$$Nu = 0.271(Pe)^{0.5} \tag{50}$$

The algebraic sum of Eq. 47 and the above equation gives the overall and average convective rate of heat transfer from all surfaces of the spherical-cap bubble as:

$$Nu = 2.113(Pe)^{0.5} \tag{51}$$

The similarity between the mass- and heat-transfer conservation equations particularly when viscous heating is ignored, allows us to write down the following equations without the need to repeat the assumptions, approximations, and solutions obtained earlier for the heat transfer:

$$Sh(\theta) = 1.693 \frac{\sin^2 \theta}{(\cos^3 \theta - 3\cos \theta + 2)^{0.5}} (ScRe)^{0.5} \quad (52)$$

for the local distribution of mass transfer over the surfaces of the spherical-cap bubble and;

$$Sh = 2.113\ (ScRe)^{0.5} \quad (53)$$

for the average mass transfer from all surfaces of the cap.

Discussion

As $E \rightarrow \infty$ in Eq. 46, the spherical cap degenerates into a circular disk and we get:

$$(Nu)_{C.D.} = 1.954(Pe)^{0.5} \quad (54)$$

for the average convective heat transfer to a circular disk. The above equation (which is considered as an asymptote to the present solution) was compared with the experimental correlation of Donaldson et al. (1971) in Figure 5 as well as with other available experimental results (Kendoush, 1994). Figure 5 shows an agreement between the asymptote of the present solution with the correlation of Donaldson et al. (1971).

A comparison was made between the ratio of Eq. 46 to that of the single sphere [that is, $(Nu)_s = 1.13\ (Pe)^{0.5}$]:

$$\frac{Nu}{(Nu)_s} = \left(\frac{3E^2+4}{E^2+4}\right)^{0.5} \quad (55)$$

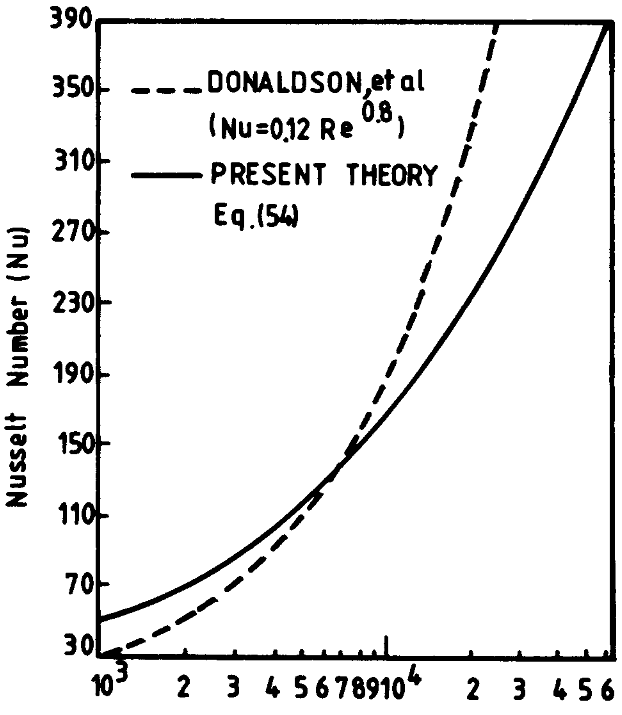

Figure 5. Agreement between asymptote of present solution with experimental correlation of Donaldson et al. (1971).

and the equivalent ratio (Eq. 6 but in heat-transfer notation) of Lochiel and Calderbank (1964) in Figure 6 where an agreement is found when $E < 1.3$ and a divergence otherwise. However, Lochiel and Calderbank's solution does not show an asymptotic behavior. Weber (1975) demonstrated that the solution of Lochiel and Calderbank (1964) gives 10–15% error when compared with the experimental data. The reason was attributed to the incorrect use of the velocity component in the steady-state mass diffusion equation.

Jean and Fan (1990) have chosen a stream function for the description of the wake of the spherical-cap bubble. This stream function is normally used for the axisymmetric plane flow towards a stagnation point. When the velocity components of this stream function was used in the mass diffusion equation, they obtained the following equation for the rear mass-transfer coefficient:

$$K_{Lr} = 2\left(\frac{6DuE}{C(E^2+4)\pi}\right)^{0.5} \text{(cm/s)} \quad (56)$$

The K_{Lr} of the present solution is obtained straightforwardly from Eq. 50 (in mass-transfer notation) as:

$$K_{Lr} = \frac{0.271\ D}{2a}\ (Sc\ Re)^{0.5} \text{ (cm/s)} \quad (57)$$

The two above equations were compared with each other in Figure 7 for constant values of θ_m, E, D, C, and a. Evidently, the stream function used by Jean and Fan (1990) was not valid for the description of vortical flow in the wake of bluff bodies; therefore, their solution did not exhibit the expected dependence of the mass-transfer coefficient on ScRe.

Equations 50 and 51 indicate that the ratio of heat or mass transfer from the concave surface to the total surface area of the spherical-cap bubble is 13 percent. Davenport et al. (1976) estimated this ratio to be 20 percent on the basis that the ratio of surface area of the frontal to the concave side is approximately 5/4 at $\theta_m = 54$. Surely, the heat-transfer ratio does not depend upon the area ratio only, as flow pattern, temperature distribution, and fluid properties are among other factors that

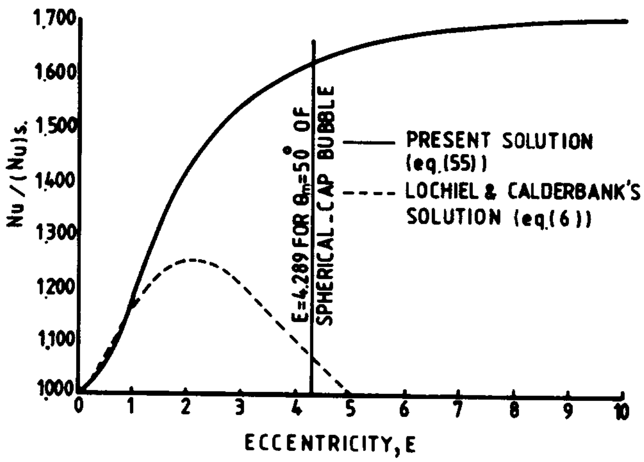

Figure 6. Comparison between present solution and Lochiel and Calderbank (1964).

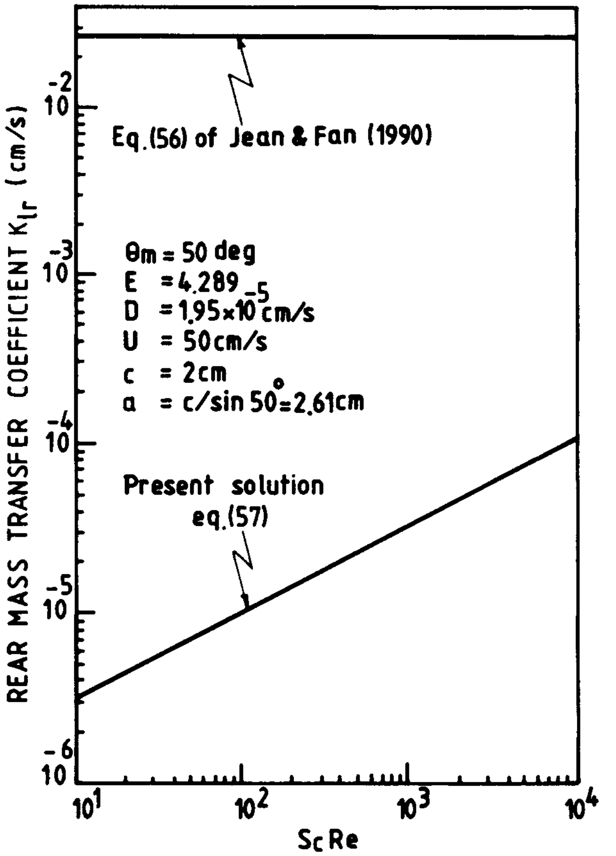

Figure 7. Comparison between present solution and Jean and Fan (1990) for the rear K_{lr}.

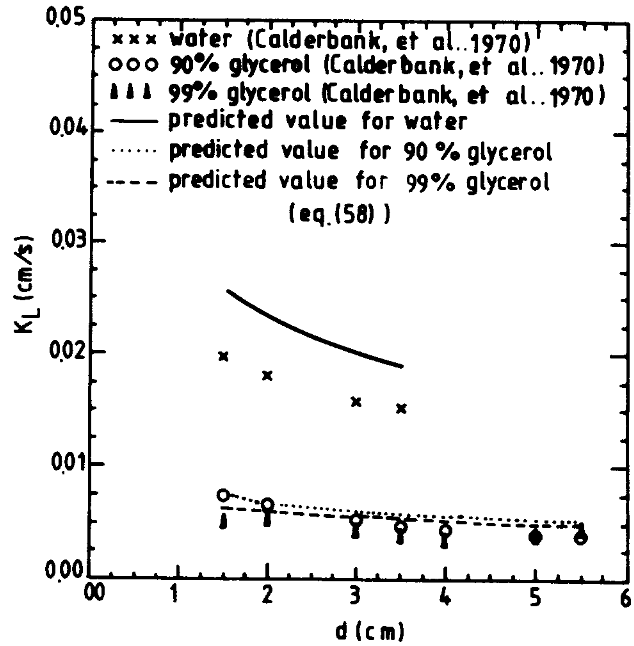

Figure 8. Comparison of present solution with experimental data of Calderbank et al. (1970)

contribute to the transfer ratio. All these factors were implicitly considered in the present analytical study.

It should be noted that the convective heat- and mass-transfer equations developed in this study are applied to the cases of high Reynolds number flows (that is, $Re \gg 45$), due to the existence of the nonlinear terms in the energy conservation equation. Specifying the above lower limit of Re is in accordance with Bhaga and Weber (1981).

The adoption of a concave surface at the rear of the spherical-cap bubble was useful for derivations of the equations. The numerical solution of Miksis et al. (1982) and the earlier experimental results of Davies and Taylor (1950) indicated the existence of such a concave surface at the rear of the cap.

In certain physical conditions, ripples appear at the rear surface of the cap (Davenport et al., 1967). However, these ripples which resemble, to a certain extent, the roughness on solid surfaces do not affect the development of the boundary layer but they only restrict the approach to the critical Reynolds number needed for the transition to turbulent boundary layer (Schlichting, 1979).

Published experimental results of Calderbank et al. (1970) were compared with the present theory (Eq. 58) as shown in Figure 8. Equation 53 with Eq. 2 for the rise velocity of the bubble were used in Figure 8 for the comparison, so that the overall mass-transfer coefficient became:

$$K_L = 1.158 d^{-1/4} D^{1/2} g^{1/4} \text{ (cm/s)} \quad (58)$$

A good agreement is obtained with the experimental data of Calderbank et al. (1970) for the glycerol in Figure 8 except for water. It is worth mentioning that the model of Jean and Fan (1990) produced exactly the same agreement for the glycerol and overestimation for water. It remains to discuss how did both the present solution and that of Jean and Fan (1990) agreed with the experimental results of Calderbank et al. (1970) in an exact manner. The solution of Jean and Fan (1990) for the frontal mass-transfer coefficient (K_L) of the cap which is essentially the same as the one derived by Lochiel and Calderbank (1964) underestimates the value of K_L in comparison with the present solution as shown implicitly in Figure 6, while the rear mass-transfer coefficient (K_{Lr}) of Jean and Fan (1990) overestimates that value as shown in Figure 7. Therefore, it is likely that these under and overestimations made the present solution and Jean and Fan's solution comparable with the experimental results of Calderbank et al. (1970) in an analogous manner.

Conclusions

The new theory developed in this article for the convective heat and mass transfer over the surfaces of the spherical-cap bubble has been successful in calculating the angular distribution of heat- and mass-transfer rates and the average transfer rates. The present analytical results were compared with the available experimental results of mass transfer. Experimental evidence concerning the rate of heat transfer over spherical-cap bubbles is sparce. With the new experimental techniques available today (for example, high spatial resolution infrared temperature detectors) this task may become achievable.

Acknowledgment

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Notation

a = radius of curvature of the spherical-cap bubble
b = half height of the spherical-cap bubble
B = constant, Eq. 1
C = basal radius of the bubble
C_D = drag coefficient, $(4\Delta\rho g d/3\rho U_\infty^2)$
d = diameter of sphere of equivalent volume to the bubble
D = diffusion coefficient
E = eccentricity, width to height ratio of spherical-cap bubble, $(2c/2b)$
E_o = Eötvos number, $(\Delta\rho g d^2/\sigma)$
g = acceleration due to gravity
h_θ = angular convective heat-transfer coefficient
H = volume of spherical-cap bubble
K = thermal conductivity
K_L = overall mass-transfer coefficient
K_{Lr} = rear mass-transfer coefficient
M = Morton number, $(g\mu^4/\rho\sigma^3)$
Nu = average Nusselt number, $[h(2a)/K]$
$(Nu)_{C.D.}$ = average Nusselt number for a circular disk
$(Nu)_s$ = average Nusselt number for a sphere
$Nu(\theta)$ = angular Nusselt number, $[h_\theta(2a)/K]$
Pe = Peclet number $[U_\infty(2a)/\alpha]$
$\dot{q}$ = rate of heat transfer per unit area of the frontal spherical surface
$\dot{q}_r$ = rate of heat transfer per unit area of the rear concave surface
$\dot{q}_\theta$ = angular rate of heat transfer per unit area
r = spherical coordinate
r_e = radius of sphere of equivalent volume to the bubble
Re = Reynolds number, $[\rho U_\infty(2a)/\mu]$
Sc = Schmidt number, (ν/D)
Sh = Sherwood number, $(2aK_L/D)$
Sh_θ = angular Sherwood number
Sh_s = Sherwood number of the sphere
T = temperature
T_a = temperature at $r=a$
u = rise velocity of spherical-cap bubble
U = tangential velocity component of the flow in spherical coordinate for the frontal spherical region of the cap
U_r = tangential velocity component of the flow in spherical coordinates for the wake region of the cap
U_∞ = uniform fluid velocity at infinity
V = radial velocity component of the flow in spherical coordinate
V_r = radial velocity component of the flow in spherical coordinates for the wake region of the cap
W = Weber number, $(\rho d U_\infty^2/\sigma)$
y = radial distance from concave surface of bubble

Greek letters

α = thermal diffusivity
δ = infinitesimal variant
δ_r = infinitesimal variant
θ = spherical coordinate
θ_m = half the angle subtended by the spherical cap bubble at the origin of the spherical envelope
λ = constant, Eq. 27
μ = dynamic viscosity of liquid
ν = kinematic viscosity of liquid
ρ = density of liquid
σ = surface tension
ϕ = variable, Eq. 31
ψ = stream function of the sphere
ψ_r = stream function of the wake of the cap
ω = variable, Eq. 30

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